General Certificate of Education
June 2006
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Monday 12 June 2006 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Two planes, Π_1 and Π_2, have equations $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = 0$ and $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = 0$ respectively.

(a) Determine the cosine of the acute angle between Π_1 and Π_2. *(4 marks)*

(b) (i) Find $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. *(2 marks)*

(ii) Find a vector equation for the line of intersection of Π_1 and Π_2. *(2 marks)*

2 A transformation is represented by the matrix $\mathbf{A} = \begin{bmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Evaluate det $\mathbf{A}$. *(1 mark)*

(b) State the invariant line of the transformation. *(1 mark)*

(c) Give a full geometrical description of this transformation. *(3 marks)*

3 Express the determinant $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ as the product of four linear factors. *(6 marks)*

4 The plane transformation T maps points (x, y) to points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(a) (i) State the line of invariant points of T. *(1 mark)*

(ii) Give a full geometrical description of T. *(2 marks)*

(b) Find $\mathbf{A}^2$, and hence give a full geometrical description of the single plane transformation given by the matrix $\mathbf{A}^2$. *(3 marks)*

5 A set of three planes is given by the system of equations

$$\begin{aligned} x &+ 3y - z = 10 \\ 2x &+ ky + z = -4 \\ 3x &+ 5y + (k-2)z = k+4 \end{aligned}$$

where k is a constant.

(a) Show that $\begin{vmatrix} 1 & 3 & -1 \\ 2 & k & 1 \\ 3 & 5 & k-2 \end{vmatrix} = k^2 - 5k + 6.$ *(2 marks)*

(b) In each of the following cases, determine the **number** of solutions of the given system of equations.

(i) $k = 1.$

(ii) $k = 2.$

(iii) $k = 3.$ *(7 marks)*

(c) Give a geometrical interpretation of the significance of each of the three results in part (b) in relation to the three planes. *(3 marks)*

6 The matrices **P** and **Q** are given by

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & t & -2 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ -7 & -1 & 5 \\ 11 & -1 & -7 \end{bmatrix}$$

where t is a real constant.

(a) Find the value of t for which **P** is singular. *(2 marks)*

(b) (i) Determine the matrix $\mathbf{R} = \mathbf{PQ}$, giving its elements in terms of t where appropriate. *(3 marks)*

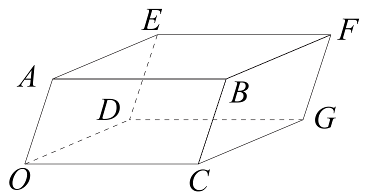
(ii) Find the value of t for which $\mathbf{R} = k\mathbf{I}$, for some integer k. *(2 marks)*

(iii) Hence find the matrix $\mathbf{Q}^{-1}$. *(1 mark)*

(c) In the case when $t = -3$, describe the geometrical transformation with matrix **R**. *(2 marks)*

Turn over for the next question

7 The diagram shows the parallelepiped $OABCDEFG$.



Points A, B, C and D have position vectors

$$\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

respectively, relative to the origin O.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(1 mark)*

(b) Determine the volume of the parallelepiped. *(3 marks)*

(c) Determine a vector equation for the plane $ABDG$:

(i) in the form $\mathbf{r} = \mathbf{u} + \lambda\mathbf{v} + \mu\mathbf{w}$; *(2 marks)*

(ii) in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(d) Find cartesian equations for the line OF, and hence find the direction cosines of this line. *(4 marks)*

8 For real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix

$$\mathbf{M} = \begin{bmatrix} a & b + a \\ b - a & -a \end{bmatrix}$$

(a) (i) Show that the eigenvalues of $\mathbf{M}$ are b and $-b$. *(3 marks)*

(ii) Show that $\begin{bmatrix} b + a \\ b - a \end{bmatrix}$ is an eigenvector of $\mathbf{M}$ with eigenvalue b. *(2 marks)*

(iii) Find an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue $-b$. *(2 marks)*

(b) By writing $\mathbf{M}$ in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix $\mathbf{D}$ and corresponding matrix $\mathbf{U}$, show that

$$\mathbf{M}^{11} = b^{10}\mathbf{M}$$ *(7 marks)*

END OF QUESTIONS